A BODY of variable mass moves about a fixed point 0 During the motion, the body continuously ejects mass The shape of the body and, consequently, its moments and products of inertia are changing due to the mass ejection

The variable-mass body is envisaged as consisting of a collection of particles of variable mass The particles continuously eject mass, but their coordinates relative to the coordinate axes attached to the body remain invariant

Mass is ejected with a nonzero velocity relative to the body axes, and consequently reactive forces are produced Thus, a torque relative to the fixed point 0 of the body is produced by the external and reactive forces acting on the body It is assumed that the ejected mass, after its separation from the body, does not affect in any way the motion of the body

Cartesian tensors[1-3] will be used in derivations Extending the summation over all particles belonging to the body at the instant t, we further define $N_i = \Sigma\epsilon_{ijk}x_jF_k$ = torque produced by the external forces, $n_i = \Sigma\epsilon_{ijk}x_j\dot{m}c_k$ = torque produced by the reactive forces, and $H_i = \Sigma\epsilon_{ijk}x_jmV_k$ = the angular momentum of body B with respect to 0

Since $V_i = \epsilon_{ijk}\omega_jx_k$, the expression for H_i can be further modified in the following way:

$$H_i = \Sigma\epsilon_{ijk}x_jmV_k = \Sigma m(x_qx_q\delta_{ij} - x_ix_j)\omega_j = I_{ij}\omega_j \quad (1)$$

where $I_{ij} = \Sigma m(x_qx_q\delta_{ij} - x_ix_j)$, inertia tensor of the variable-mass body B

Generally, during the mass ejection, the inertia tensor may change Therefore, differentiating the angular momentum with respect to the body axes,

$$\frac{\delta H_i}{\delta t} = \frac{\delta I_{ij}}{\delta t}\,\omega_j + I_{ij}\frac{\delta\omega_j}{\delta t}, \qquad \frac{\delta I_{ij}}{\delta t} \neq 0 \quad (2)$$

On the other hand, differentiating $H_i = \Sigma\epsilon_{ijk}x_jmV_k$ with respect to the fixed coordinate system, we obtain

$$dH_i/dt = \Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) \quad (3)$$

These are related by the standard equation $dH_i/dt = \delta H_i/\delta t + \epsilon_{ijk}\omega_jH_k$, giving

$$(\delta H_i/\delta t) + \epsilon_{ijk}\omega_jH_k = \Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) \quad (4)$$

In order to express the right member of (4) in a new form, we consider the equation of motion for a single particle of variable mass m as[4]

$$m(du_i/dt) = F_i + \dot{m}c_i \quad (5)$$

Adding $\dot{m}v_i$ to both sides of (5), we obtain

$$(d/dt)(mV_i) = F_i + \dot{m}u_i \quad (6)$$

Forming a cross product of x_i and (6), and summing over all particles of the system, we obtain

$$\Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) = \Sigma\epsilon_{ijk}x_jF_k + \Sigma\epsilon_{ijk}x_j\dot{m}u_k \quad (7)$$

Substituting $u_i = c_i + V_i$ and $V_i = \epsilon_{ijk}\omega_jx_k$ into the right member of (7), we obtain

$$\Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) = N_i + n_i + \Sigma\epsilon_{ijk}x_j\dot{m}(\epsilon_{kpq}w_px_q) \quad (8)$$

The last term of (8) can be further expanded into the form

$$\Sigma\epsilon_{ijk}x_j\dot{m}\epsilon_{kpq}\omega_px_q = \Sigma\dot{m}(x_qx_q\delta_{ij} - x_ix_j)\omega_j \quad (9)$$

Now $\dot{m} = dm/dt = \delta m/\delta t$, since the rate of mass ejection is the same when measured by observers stationed in fixed or moving coordinate systems Equation (9) can be modified to the form

$$\Sigma\epsilon_{ijk}x_j\dot{m}\epsilon_{kpq}\omega_px_q = \Sigma(\delta m/\delta t)(x_qx_q\delta_{ij} - x_ix_j)\omega_j = \omega_j(\delta/\delta t)[m(x_qx_q\delta_{ij} - x_ix_j)] = (\delta I_{ij}/\delta t)\omega_j \quad (10)$$

since coordinates x_i of particles m are constants with respect to the body axes Combining Eqs (1, 2, 4, 8, and 10), we obtain Euler's dynamical equation for the body of variable mass moving about a fixed point:

$$I_{ij}(\delta\omega_j/\delta t) + \epsilon_{ijk}\omega_jI_{kq}\omega_q = N_i + n_i \quad (11)$$

An interesting feature of Eq (11) is the fact that $\delta I_{ij}/\delta t$ does not appear

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Extension of Lifting-Line Theory to a Cascade of Split Aerofoils

B LAKSHMINARAYANA*

Pennsylvania State University, University Park, Pa

Nomenclature

ω	=	potential function
Γ	=	circulation
S	=	spacing of the aerofoils
y,z	=	Cartesian coordinates
u	=	induced velocity at the lifting line
$2l$	=	total length of the aerofoil
2τ	=	gap width
B_1,B_2 B_n	=	Fourier coefficients
V	=	inlet velocity
$\Omega = \tau/l$	=	gap/length ratio
a_0	=	slope of the lift angle-of-attack curve
C	=	chord length
β'	=	angle of attack measured from the attitude of no lift
L	=	total lift
D_i	=	induced drag
C_{Di}	=	induced drag coefficient
ρ	=	fluid density
μ	=	$a_0C/8l$

ANALYTICAL and experimental investigation on the effect of a gap in a cascade aerofoil is investigated in this note Using lifting-line theory, expressions are given for induced velocity and drag of such an aerofoil The type of analysis presented has its application in predicting the tip clearance losses in axial flow turbomachines [1]

Analysis

When a gap is cut in a cascade aerofoil, the bound vortices cannot cross the gap and must, therefore, be shed off as trailing vortices Figure 1 shows the trailing vortices, and also the trailing vortex system for a cascade of split aerofoils

The potential function of a row of vortices that are infinite in one direction along the x axis (Fig 1b) is given by Milne-Thompson[2] as

$$\omega = (i\Gamma_{\text{shed}}/4S)\log\sin[\pi(y + iz)/S] \quad (1)$$

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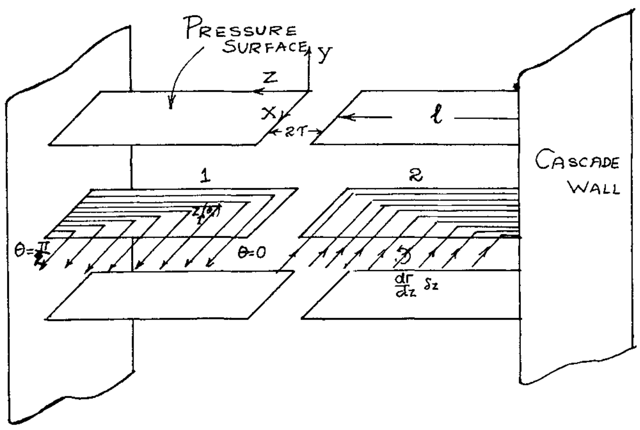
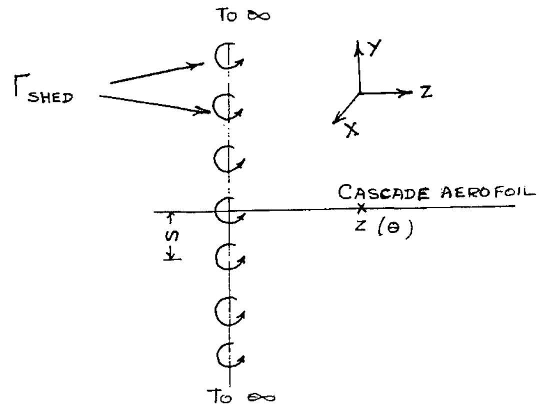
* Visiting Assistant Professor of Aeronautical Engineering, Department of Aeronautical Engineering Member AIAA



a) **Trailing vortices along the cascade aerofoil**



b) **A row of vortices**

Fig 1 Trailing vortex system in a cascade of split aerofoils

At the lifting line of the center aerofoil $y = 0$, and hence the induced velocity perpendicular to lifting line is

$$du = (-d\omega/dz) = (-\Gamma_{\text{h d}}/4S)\coth(\pi z/S) \quad (2)$$

Let the circulation distribution along the aerofoil be given by[3]

$$\Gamma = 4lV\Sigma B_n \quad \sin n\theta \quad (3)$$

where z, the spanwise coordinate, is given by

$$\left.\begin{aligned} z &= l(1 - \cos\theta) && 0 \leqslant \theta \leqslant \pi/2 \text{ for blade 1} \\ z &= l(1 + 2\Omega - \cos\theta) && 0 \leqslant \theta \leqslant \pi/2 \text{ for blade 2} \end{aligned}\right\} \quad (4)$$

Using Eqs (2–4), and knowing that

$$\Gamma_{\text{sh d}} = (d\Gamma/dz)\ \delta z \quad (5)$$

it can be proved that the total velocity induced perpendicular to the lifting line at any point $z_1(\theta_1)$ is

$$u = \frac{Vl}{S}\left[\int_0^{\pi/2} \sum_{n=1\,2}^{\infty} nB_n \cos n\theta \ \coth\frac{\pi l}{s} \times (\cos\theta - \cos\theta_1)d\theta - \int_0^{\pi/2} \sum_{n=1\,2}^{\infty} nB_n \times \cos n\theta \ \coth\frac{\pi l}{s}\ (2 + 2\Omega - \cos\theta - \cos\theta_1)d\theta\right] \quad (6)$$

The first integral in Eq (6) presents certain difficulties due to singularity at $\theta = \theta_1$ The difficulty could be overcome in the following way It can be proved that

$$\int_0^{\pi/2} \cos n\theta \ \coth\frac{\pi l}{s}\ (\cos\theta - \cos\theta_1)d\theta = \int_0^{\pi/2} \cos n\theta \ d\theta + \int_0^{\pi/2} \frac{2\cos n\theta \ \exp(2\pi l \cos\theta_1/s)}{\exp(2\pi l\cos\theta/s) - \exp(2\pi l\cos\theta_1/s)}\, d\theta \quad (7)$$

The second integral in Eq (7) can be simplified by expanding the numerator and denominator of the integral into a series by Taylor's theorem By so doing, it can be proved that

$$\int_0^{\pi/2} \cos n\theta \ \coth\frac{\pi l}{s}\ (\cos\theta - \cos\theta_1)d\theta = \sum_{n=1\,2}^{\infty} \frac{s}{\pi l \sin\theta_1}\left[-n\cos n\theta_1 \log\left(\frac{\pi/2 - \theta_1}{\theta_1}\right) + \pi\left\{n \sin n\theta_1 + \left(\frac{\cos\theta_1}{2} - K\sin\theta_1\right)\cos n\theta_1\right\} + \frac{\pi}{24}\left(\frac{\pi}{2} - 2\theta_1\right)(1 + 2K\cos\theta_1 + 4\cos\theta_1 + \sin^2\theta_1) + \frac{1}{3}\left\{\left(\frac{\pi}{2} - \theta_1\right)^3 + \theta_1^3\right\}\{\quad \text{etc}\ \}\right] \quad (8)$$

where $K = (\pi l/S)$

It is evident that the analytical solution is nonconvergent and unsuitable for evaluating the numerical values

Therefore Eq (6) has been used in preference to Eq (8) in all the foregoing analyses and numerical calculations on the Liverpool University DEUCE computer

By substituting for Γ, a_0, and u in the fundamental equation connecting the circulation and lift,

$$\Gamma = (a_0C/2)V[\beta' - (u/V)] \quad (9)$$

it can be proved that

$$\Sigma B_n \sin n\theta_1 + \mu\frac{l}{s}\left[\int_0^{\pi/2} \sum_{n=1\,2}^{\infty} nB_n \cos n\theta \times \left\{\coth\pi l\,\frac{(\cos\theta - \cos\theta_1)}{s} - \coth\frac{\pi l}{s}\ (2 + 2\Omega - \cos\theta - \cos\theta_1)\right\} d\theta\right] = \mu\beta' \quad (10)$$

Three† such equations are obtained from the three spanwise positions, and the three coefficients may then be determined Knowing the values of these coefficients, the induced velocity can then be determined from Eq (6)

Induced Drag

Knowing that

$$L = 2\int_0^{\pi/2} \rho V_1 \Gamma dz = 2\pi\rho V_1^2 l^2 B_1 \quad (11)$$

and

$$D_i = 2\int_0^{l} \rho u \Gamma dz \quad (12)$$

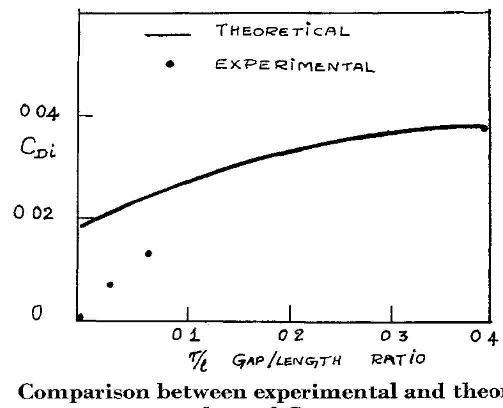


Fig 2 Comparison between experimental and theoretical values of C_{Di}

† Since successive coefficients B_1, B_2, B_n decrease rapidly in magnitude, only three coefficients have been retained in the foregoing calculations

it can be proved by substitution and simplification that

$$C_{Di} = 8\frac{l}{s}\frac{l}{c}\int_0^{\pi/2}\sum_{n=1\,2}^{\infty} B_n \sin n\theta_1 \sin\theta_1 \times \left\{\int_0^{\pi/2}\sum_{n=1\,2}^{\infty} nB_n \cos n\theta \coth\frac{\pi l}{s}(\cos\theta - \cos\theta_1)d\theta - \int_0^{\pi/2}\sum_{n=1\,2}^{\infty} nB_n\cos n\theta \coth\frac{\pi l}{s} \times (2 + 2\Omega - \cos\theta - \cos\theta_1)d\theta\right\} d\theta_1 \quad (13)$$

Numerical Results and Experimental Verification

A cascade of split compressor blades (10C4 30 C50-British Profile), whose lift angle-of-attack characteristic is known, was selected, and measurements of lift and induced drag were carried out at Reynolds number (with respect to chord) of 2×10^5 The results so obtained are plotted in Fig 2

Liverpool University DEUCE Computer has been used to evaluate induced drag coefficients theoretically, and the results are plotted and compared with experimental values in Fig 2

Discussion and Conclusions

The agreement between theoretical and experimental values is reasonably good at high gap/chord ratios At low gap/chord ratios, the real fluid effects inside the gap, the finite thickness, and chord length of the blade reduce leakage flow and, hence, the strength of the trailing vortices This accounts for the poor agreement between the theoretical and experimental values at low gap/chord ratios A modified analysis has been suggested for low gap/chord ratios, and this has been dealt with in Ref 1

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Location of Catch-Up Point in a ΔV Perturbed Circular Orbit

Andrew H Milstead*

Aerospace Corporation, El Segundo, Calif

Introduction

Reference 1 compares the position of a drag-free satellite in a circular orbit with the position of a drag-perturbed satellite that was initially in the same circular orbit (and at the same position) as the drag-free body At the time when the drag is "turned-on," the drag-perturbed satellite departs from the circular orbit and initially assumes a smaller angular velocity than the drag-free satellite, thereby falling behind the drag-free satellite in angular position Because of the conservation of angular momentum, the angular velocity of the drag-perturbed satellite increases as its altitude decreases, and it overtakes the drag-free satellite in angular position The angular position (measured from the position where the drag is turned on) at which the drag-perturbed satellite overtakes the drag-free satellite has been shown in Ref 1 to be independent of the drag force to first order and has a value of 0 292 revolutions $\approx 105°$

The purpose of this note is to find the "catch-up" angular position when the perturbing force is an instantaneous velocity addition (ΔV) applied parallel and opposite in direction to the circular velocity Results show that the catch-up angular position (measured from the point of application of the ΔV) varies between 63 640° and 73 092° as the ΔV varies from circular velocity to zero

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* Member of the Technical Staff Member AIAA

Analysis

This analysis will define the catch-up point for the limiting cases where ΔV approaches zero and where ΔV approaches retrograde circular velocity The method of calculating the catch-up point for other values of ΔV will be indicated

If two bodies are traveling together in a circular orbit in an inverse-square force field, and one of the bodies is perturbed by a ΔV parallel to its velocity vector, that body will enter an ellipse (assuming ΔV is not sufficient to cause escape) that is tangent to the original circle One apse of the ellipse will be at a distance r (the original circular orbit radius) from the force center, and the other apse will be at a distance of $r \pm \Delta r$, where + implies a ΔV along the circular velocity vector, and − implies a ΔV opposite to the circular velocity vector The semimajor axis and eccentricity for the new ellipse are given, respectively, by

$$a = r[1 \pm \chi] \qquad e = \chi/[1 \pm \chi] \quad (1)$$

where $\chi = \Delta r/2r$ From the vis-viva equation,

$$\frac{\Delta V}{V} = \left|1 - \left(\frac{1 \pm 2\chi}{1 \pm \chi}\right)^{1/2}\right| \quad (2)$$

If the ΔV is applied to the circular velocity in a retrosense, the minus signs apply in Eqs (1) and (2) Let θ_E be the angular travel of the body in the ellipse, and let θ be the angular travel of the body in the circle, both measured from the point of application of the retrograde ΔV The true anomaly of the body in the ellipse is then

$$v = \pi + \theta_E \quad (3)$$

and the mean anomaly is

$$M = \frac{\mu^{1/2}}{a^{3/2}}\left[\frac{\pi a^{3/2}}{\mu^{1/2}} + \frac{r^{3/2}\theta_c}{\mu^{1/2}}\right] \quad (4)$$

where

μ = gravitational constant
$\mu^{1/2}/a^{3/2}$ = mean motion of the body in the ellipse
$\pi a^{3/2}/\mu^{1/2}$ = half of the period of the body in the ellipse
$(r^{3/2}/\mu^{1/2})\theta$ = time measured from apogee

Using Eq (1) in (4) and simplifying gives

$$M = \pi + [\theta/(1-\chi)^{3/2}] \quad (5)$$

Reference 2 (p 171) gives the relation between v and M in an ellipse as

$$v = M + 2e\sin M + \tfrac{5}{4}e^2 \sin 2M + 0(e^3) \quad (6)$$

Substituting (1, 3, and 5) into (6) and expanding in power of χ gives

$$\pi + \theta_E = \pi + \theta[1 + \tfrac{3}{2}\chi + 0(\chi^2)] + 2\chi[1 + \chi + 0(\chi^2)]\sin\{\pi + \theta[1 + \tfrac{3}{2}\chi + 0(\chi^2)]\} + 0(\chi^2)$$

Now, setting $\theta_E = \theta$ and dividing through by χ yields

$$0 = \theta[\tfrac{3}{2} + 0(\chi)] - 2[1 + 0(\chi)]\sin\{\theta[1 + 0(\chi)]\} + 0(\chi)$$